

Differential equations defined on algebraic curves

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Abstract. The class of ordinary linear constant coefficient differential equations is naturally embedded into a wider class by associating differential equations to algebraic curves.

Key words. Convergent formal series, algebraic curves, differential equations, Bessel functions.

Mathematics Subject Classification: 14H05, 34A09

Introduction

Throughout, \mathbb{F} is the field of real or complex numbers, T is an indeterminate, and $\mathbb{F}\{T\}$ denotes the domain of convergent formal series in T with coefficients in the field \mathbb{F} . (We remind that a formal series $\sum_{n \geq 0} a_n T^n$ is said to be convergent, if there exists a positive real number r such that $\sum_{n \geq 0} |a_n| r^n < +\infty$.) The fraction field of the domain $\mathbb{F}\{T\}$ is denoted by $\mathbb{F}(\{T\})$. Elements of this field are referred to as convergent Laurent series. By the theorem on units (see Section 4.4 in Remmert [1]), stating that an element of the ring $\mathbb{F}\{T\}$ is invertible if and only if its free term $\neq 0$, it is immediate that every convergent Laurent series can be written as a fraction f/T^m , where $f \in \mathbb{F}\{T\}$ and $m \geq 0$.

Throughout, I is an interval of real axis (with non-empty interior) on which a point is fixed. (The point of this latter is that it permits to define canonically the indefinite integral. In applications, it is customary to regard an interval as a time axis and a fixed point on it as an initial time.) Without loss of generality, we certainly may assume that $0 \in I$ and that this zero is chosen as a fixed point. We let $C(I)$ denote the space of all \mathbb{F} -valued continuous functions defined on the interval I and $\mathcal{D}'(I)$ the space of all \mathbb{F} -valued Schwartz distributions.

The space $C(I)$ has a natural structure of a module over $\mathbb{F}\{T\}$. (The " T " acts on continuous functions as the integral operator, and this action is extended in a unique way to all convergent formal series by linearity and continuity.) So that, for every $f \in \mathbb{F}\{T\}$, we have an operator

$$C(I) \xrightarrow{f} C(I), \quad w \mapsto fw.$$

On the other hand, for every $m \geq 0$, there is a differential operator $D^m : \mathcal{D}'(I) \rightarrow \mathcal{D}'(I)$. Restricting it to $C(I)$, we get an operator

$$C(I) \xrightarrow{D^m} \mathcal{D}'(I).$$

Given a convergent Laurent series $\phi = f/T^m$ with $f \in \mathbb{F}\{T\}$ and $m \geq 0$, define D_ϕ to be the composition

$$C(I) \xrightarrow{f} C(I) \xrightarrow{D^m} \mathcal{D}'(I).$$

A simple observation is that ordinary linear constant coefficient differential operators have the form D_ϕ . Indeed, if $f \in \mathbb{F}[s]$, then the operator

$$f(D) : C(I) \rightarrow \mathcal{D}'(I)$$

can be written as

$$f(D) = D_\phi,$$

where $\phi = f(T^{-1})$. Next, as is well-known, one may view f as a rational function on the projective line \mathbb{P}^1 associated with $\mathbb{F}[s]$. Notice also that ϕ is the image of f under the embedding of $\mathbb{F}(s)$ (which is the rational function field of \mathbb{P}^1) into $\mathbb{F}(\{T\})$ that takes s^{-1} (which is a local parameter at the infinite point) to T .

The theory of linear constant coefficient differential equations can be easily generalized to the following situation. We replace $(\mathbb{P}^1, \infty, s^{-1})$ by the triple (X, P, t) , where X is an irreducible smooth projective algebraic curve, P a rational point on X and t a local parameter of X at this point. The pair (P, t) determines a canonical embedding

$$\sharp : \mathbb{F}(X) \rightarrow \mathbb{F}(\{T\}),$$

where $\mathbb{F}(X)$ is the rational function field of X . For every $f \in \mathbb{F}(X)$, we define the operator $f(D) : C(I) \rightarrow \mathcal{D}'(I)$ by setting

$$f(D) = D_{\sharp(f)}.$$

It is our belief that differential equations of the form

$$f(D)w = 0$$

may be a source for many interesting functions.

To demonstrate that the generalization is meaningful, we show that well-known Bessel functions are solutions of some differential equations on the hyperbola defined by the equation $y^2 - x^2 = 1$. More precisely, we show that, for each nonnegative integer n , the Bessel function J_n is a solution of the differential equation associated with the rational function $y(x + y)^n$ and that this is the unique solution that satisfies the following initial conditions

$$w(0) = w'(0) = \cdots = w^{(n-1)}(0) = 0 \quad \text{and} \quad w^{(n)}(0) = \frac{1}{2^n}.$$

Moreover, we shall see that

$$J_n, D(J_n), \dots, D^n(J_n)$$

constitute a fundamental system of solutions.

1 Differential equations associated with algebraic curves

For every continuous function $w \in C(I)$, let

$$\int w$$

denote the "normalized" indefinite integral of w defined by

$$(\int w)(\xi) = \int_0^\xi w(\alpha) d\alpha, \quad \xi \in I.$$

Given $f \in \mathbb{F}\{T\}$ and $w \in C(I)$, define the product fw by the formula

$$fw = \sum_{n \geq 0} a_n \int^n w,$$

where a_n are the coefficients of f . (The series converges uniformly on every compact neighborhood of 0.) This multiplication makes $C(I)$ a *module* over $\mathbb{F}\{T\}$. It is worth noting that

$$1w = w \quad \text{and} \quad Tw = \int w.$$

Define the E -transform $E(f)$ of the convergent formal series $f = \sum_{n \geq 0} a_n T^n$ as the entire analytic function

$$\xi \mapsto \sum_{n \geq 0} a_n \xi^n / n! \quad (\xi \in I).$$

Remark that $E(f) = f\mathbf{1}$, where $\mathbf{1}$ denotes the constant function that is identically 1 on I .

Lemma 1 *Let $f, g \in \mathbb{F}\{T\}$ and $m, n \geq 0$. If $T^n f = T^m g$, then the two compositions*

$$C(I) \xrightarrow{f} C(I) \xrightarrow{D^m} \mathcal{D}'(I) \quad \text{and} \quad C(I) \xrightarrow{g} C(I) \xrightarrow{D^n} \mathcal{D}'(I)$$

are equal to each other.

Proof. This is obvious. Indeed,

$$\forall w \in C(I), \quad D^m(fw) = D^{m+n}(T^n fw) = D^{m+n}(T^m gw) = D^n(gw).$$

□

If f/T^m and g/T^n are two representations of the same convergent Laurent series, then $T^n f = T^m g$. In view of the above lemma, it is natural therefore to make the following definition.

Definition. Let $\phi \in \mathbb{F}(\{T\})$, and assume that $\phi = f/T^m$ with $f \in \mathbb{F}\{T\}$ and $m \geq 0$. Define the differential operator

$$D_\phi : C(I) \rightarrow \mathcal{D}'(I)$$

to be the composition

$$C(I) \xrightarrow{f} C(I) \xrightarrow{D^m} \mathcal{D}'(I).$$

Every $\phi \in \mathbb{F}(\{T\})$ can be uniquely written in the form

$$\phi = \sum_{n > -\infty} a_n T^n.$$

(The coefficients a_n are zero for all but finitely many negative values of n .) The least integer n for which $a_n \neq 0$ is called the order ϕ . (In case when $\phi = 0$ the order is defined to be $+\infty$.) Define the degree of the differential equation

$$D_\phi w = 0$$

as minus the order of ϕ . If the degree is non-positive, i.e., if $\phi \in \mathbb{F}\{T\}$, then the operator D_ϕ is injective, and consequently the differential equation has no solutions other than 0. Of interest may be only equations of positive degree, i.e., those equations that correspond to convergent Laurent series of the form u/T^n , where u is a unit in $\mathbb{F}\{T\}$ and n is a positive integer.

Let $\mathbb{F}[T]_{\leq k}$ denote the space of polynomials (in T) of degree $\leq k$.

Theorem 1 *If $\phi \in \mathbb{F}(\{T\})$ and if $\phi = u/T^n$ with unit u and $n \geq 1$, then the equation*

$$D_\phi w = 0$$

has solutions

$$w = E\left(\frac{p}{u}\right), \quad p \in \mathbb{F}[T]_{\leq n-1}.$$

Proof. We have:

$$D_\phi w = 0 \Leftrightarrow D^n(uw) = 0.$$

One knows well that the kernel of the operator

$$D^n : C(I) \rightarrow \mathcal{D}'(I)$$

is the space of polynomial functions of degree $\leq n-1$, i.e., the space $E(\mathbb{F}[T]_{\leq n-1})$. Because u is invertible, the operator $C(I) \xrightarrow{u} C(I)$ is bijective. Hence,

$$D^n(uw) = 0 \Leftrightarrow uw \in E(\mathbb{F}[T]_{\leq n-1}) \Leftrightarrow w \in E(u^{-1}\mathbb{F}[T]_{\leq n-1}).$$

The proof is complete. \square

Of particular interest must be differential equations associated with those convergent Laurent series that come from algebraic functions, i.e., rational functions on algebraic curves.

Assume we have a triple (X, P, t) , where X is an irreducible smooth projective algebraic curve, P a rational point on X and t a local parameter of X at this point. Letting O_P denote the local ring of P and $\mathbb{F}[[t]]$ the ring of formal series in t , we have

$$O_P \subseteq \mathbb{F}[[t]].$$

In fact, by the implicit function theorem, $O_P \subseteq \mathbb{F}\{t\}$. Taking t to T , we get a canonical one-to-one ring homomorphism

$$O_P \rightarrow \mathbb{F}\{T\},$$

which, in turn, induces an embedding

$$\mathbb{F}(X) \rightarrow \mathbb{F}(\{T\}),$$

where $\mathbb{F}(X)$ stands for the rational function field of X . Denote this canonical embedding by \sharp .

Definition. Given a rational function $f \in \mathbb{F}(X)$, define a "linear constant coefficient differential" operator $f(D)$ to be $D_{\sharp(f)}$.

We close the section with two examples.

Example 1. Let X be a projective line, P any its rational point and t a local parameter at P such that $s = t^{-1}$ is regular everywhere outside of P . Let \sharp be the embedding of $\mathbb{F}(s) = \mathbb{F}(t)$ into $\mathbb{F}(\{T\})$ for which $\sharp(t) = T$.

An ordinary linear constant coefficient differential equation

$$a_0 w^{(n)} + a_1 w^{(n-1)} + \cdots + a_n w = 0$$

with $a_0 \neq 0$ can be rewritten as

$$f(D)w = 0,$$

where $f = a_0 s^n + a_1 s^{n-1} + \cdots + a_n$. We have

$$\sharp(f) = T^{-n}(a_0 + a_1 T + \cdots + a_n T^n).$$

Consequently, by Theorem 1, the solutions of this equation are

$$w = E\left(\frac{p}{a_0 + a_1 T + \cdots + a_n T^n}\right), \quad p \in \mathbb{F}[T]_{\leq n-1}.$$

Example 2. Let n be a nonnegative integer. Recall that the Laguerre polynomial of degree n is defined by the formula

$$L_n(\xi) = \frac{e^\xi}{n!} \left(\frac{d}{d\xi}\right)^n (\xi^n e^{-\xi}).$$

Using the Leibniz rule, one has

$$L_n(\xi) = \frac{e^\xi}{n!} \sum \binom{n}{k} ((-1)^k e^{-\xi}) \left(\frac{n!}{k!} \xi^k\right) = \sum \binom{n}{k} \frac{(-1)^k}{k!} \xi^k,$$

and thus $L_n = E((1 - T)^n)$.

Let now (X, P, t) and \sharp be as in the previous example. Take the rational function $f = s^{n+1}/(s - 1)^n$. Because

$$\sharp(s^{n+1}/(s - 1)^n) = \frac{T^{-n-1}}{(T^{-1} - 1)^n} = T^{-1}(1 - T)^{-n},$$

the solutions of $f(D)w = 0$ are

$$w = cE((1 - T)^n) = cL_n, \quad c \in \mathbb{F}.$$

2 Bessel functions as special functions associated with hyperbola

Let us consider the hyperbola defined by the equation

$$Y^2 - X^2 = Z^2.$$

In the affine coordinates $x = X/Z$ and $y = Y/Z$ this is given by

$$y^2 - x^2 = 1.$$

The hyperbola has two points at infinity, namely, $(1 : 1 : 0)$ and $(1 : -1 : 0)$. Take $P = (1 : 1 : 0)$, say. In the affine piece $X \neq 0$, the affine coordinates are $u = Y/X$ and $t = Z/X$, so that the equation takes the form $u^2 - 1 = t^2$ and our infinite point becomes $(1, 0)$. Choose t as a local parameter at P , and let \sharp denote the canonical embedding

$$\mathbb{F}(x, y) \rightarrow \mathbb{F}\{T\}$$

determined by the pair (P, t) . By the implicit function theorem,

$$u = \sqrt{1 + t^2} = 1 + \frac{1}{2}t^2 + \frac{1}{2}\left(\frac{1}{2} - 1\right)\frac{t^4}{2!} + \frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)\frac{t^6}{3!} + \dots$$

near the point P .

Since $y = t^{-1}u$, we have $y = t^{-1}\sqrt{1 + t^2}$. For each $n \geq 0$, let us set

$$j_n = y(x + y)^n.$$

Letting $\sqrt{1 + T^2}$ denote the convergent formal series

$$1 + \frac{1}{2}T^2 + \frac{1}{2}\left(\frac{1}{2} - 1\right)\frac{T^4}{2!} + \frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)\frac{T^6}{3!} + \dots,$$

we have

$$\sharp(j_n) = T^{-(n+1)}\sqrt{1 + T^2}(1 + \sqrt{1 + T^2})^n.$$

Remark that $\sqrt{1 + T^2}(1 + \sqrt{1 + T^2})^n$ is a unit in $\mathbb{F}\{T\}$ and its free term is equal to 2^n . Hence, the equation $j_n(D)w = 0$ has degree $n + 1$ and, by Theorem 1, its solutions are

$$w = E\left(\frac{p}{\sqrt{1 + T^2}(1 + \sqrt{1 + T^2})^n}\right), \quad p \in \mathbb{F}[T]_{\leq n}.$$

Put

$$J_n = E\left(\frac{T^n}{\sqrt{1 + T^2}(1 + \sqrt{1 + T^2})^n}\right).$$

One sees that this is the solution with initial conditions

$$w(0) = w'(0) = \dots w^{(n-1)}(0) = 0 \quad \text{and} \quad w^{(n)}(0) = \frac{1}{2^n}.$$

One can see also that

$$J_n, D(J_n), \dots, D^n(J_n)$$

constitute a fundamental system of solutions.

We are going to show that the functions J_n are none other than the Bessel functions.

Lemma 2 *We have:*

$$J_0(\xi) = 1 - \frac{\xi^2}{4} + \frac{\xi^4}{(2!)^2 4^2} - \frac{\xi^6}{(3!)^2 4^3} + \dots$$

and

$$J_1(\xi) = \frac{\xi}{2} - \frac{\xi^3}{2 \cdot 2^3} + \frac{\xi^5}{2! \cdot 3! \cdot 2^5} - \frac{\xi^7}{3! \cdot 4! \cdot 2^7} + \dots$$

Proof. Using the equality

$$1 \cdot 3 \cdot \dots \cdot (2k-1) = \frac{(2k)!}{k! 2^k},$$

we find that

$$\begin{aligned} \frac{1}{\sqrt{1+T^2}} &= \sum_{k \geq 0} \frac{(-\frac{1}{2})(-\frac{1}{2}-1) \dots (-\frac{1}{2}-k+1)}{k!} T^{2k} = \\ &= \sum_{k \geq 0} \frac{(-1)^k \cdot 1 \cdot 3 \cdot \dots \cdot (2k-1)}{2^k k!} T^{2k} = \sum_{k \geq 0} \frac{(-1)^k (2k)!}{4^k (k!)^2} T^{2k}. \end{aligned}$$

It follows that

$$J_0(\xi) = E\left(\frac{1}{\sqrt{1+T^2}}\right) = \sum_{k \geq 0} \frac{(-1)^k}{4^k (k!)^2} \xi^{2k}.$$

Next, we have

$$\begin{aligned} \frac{T}{\sqrt{1+T^2}(1+\sqrt{1+T^2})} &= \frac{\sqrt{1+T^2}-1}{T\sqrt{1+T^2}} = T^{-1}\left(1 - \frac{1}{\sqrt{1+T^2}}\right) = \\ &= \sum_{k \geq 1} \frac{(-1)^{k+1} 1 \cdot 3 \cdot \dots \cdot (2k-1)}{2^k k!} T^{2k-1} = \sum_{k \geq 1} \frac{(-1)^{k+1} (2k)!}{2^{2k} k! k!} T^{2k-1} = \sum_{k \geq 1} \frac{(-1)^{k+1} (2k-1)!}{2^{2k-1} (k-1)! k!} T^{2k-1}. \end{aligned}$$

It follows that

$$J_1(\xi) = E\left(\frac{T}{\sqrt{1+T^2}(1+\sqrt{1+T^2})}\right) = \sum_{k \geq 1} \frac{(-1)^{k+1}}{2^{2k-1} (k-1)! k!} \xi^{2k-1}.$$

The proof is complete. \square

Lemma 3 *For every $n \geq 1$, we have:*

$$2J'_n = J_{n-1} - J_{n+1}.$$

Proof. For $n \geq 0$, put

$$g_n = \frac{T^n}{\sqrt{1+T^2}(1+\sqrt{1+T^2})^n}.$$

We have

$$2(1 + \sqrt{1+T^2}) = (1 + \sqrt{1+T^2})^2 - T^2.$$

Multiplying this by $T^{-1}g_{n+1}$, we get

$$2T^{-1}g_n = g_{n-1} - g_{n+1}.$$

Applying the E -transform, we complete the proof. \square

It immediately follows from the above two lemmas that J_n are Bessel functions (of first kind).

Thus, we have proved the following theorem.

Theorem 2 *Let n be a nonnegative integer. A fundamental system of solutions of the differential equation*

$$j_n(D)w = 0$$

is

$$J_n, D(J_n), \dots, D^n(J_n),$$

where J_n is the Bessel function of order n . Moreover, J_n is the solution that satisfies the following initial conditions

$$w(0) = w'(0) = \dots = w^{(n-1)}(0) = 0 \quad \text{and} \quad w^{(n)}(0) = \frac{1}{2^n}.$$

References

- [1] R. Remmert, Theory of Complex Functions, Springer-Verlag, New York, 1991.